Received: *April 7,2000*, Accepted: *July 7,2000*

On the Landau-Ginzburg description of boundary CFTs and special lagrangian submanifolds

Suresh Govindarajan

Department of Physics
Indian Institute of Technology, Madras
Chennai 600 036 INDIA
Email: suresh@chaos.iitm.ernet.in

T. Jayaraman

The Institute of Mathematical Sciences
C.I.T. Campus, Taramani
Chennai 600 113 INDIA
Email: jayaram@imsc.ernet.in

Abstract: We consider Landau-Ginzburg (LG) models with boundary conditions preserving A-type $N = 2$ supersymmetry. We show the equivalence of a linear class of boundary conditions in the LG model to a particular class of boundary states in the corresponding CFT by an explicit computation of the open-string Witten index in the LG model. We extend the linear class of boundary conditions to general non-linear boundary conditions and determine their consistency with A-type $N = 2$ supersymmetry. This enables us to provide a microscopic description of special lagrangian submanifolds in $\mathbb{C}^n$ due to Harvey and Lawson. We generalise this construction to the case of hypersurfaces in $\mathbb{P}^n$. We find that the boundary conditions must necessarily have vanishing Poisson bracket with the combination $(W(\phi) - \overline{W}(\overline{\phi}))$, where $W(\phi)$ is the appropriate superpotential for the hypersurface. An interesting application considered is the $\mathbb{T}^3$ supersymmetric cycle of the quintic in the large complex structure limit.

Keywords: Field Theories in Lower Dimensions, D-branes, Conformal Field Models in String Theory.

JHEP07(2000)016

Contents

1. Introduction

A complete microscopic description of D-branes wrapped on supersymmetric cycles is available in the the cases where these cycles are submanifolds in flat spaces like tori. The description can also be fairly reliably extended to spaces where the techniques of conformal field theories constructed from purely free fields can be easily applied, as in the case of orbifolds. However it is only recently that the case of D-branes living in non-trivial curved spaces and wrapped on supersymmetric cycles in these spaces have begun to be investigated systematically from a microscopic viewpoint. Following Ooguri et al. [1], who specified the boundary conditions on the worldsheet $N = 2$ supersymmetry generators and explained their geometric significance, further efforts have concentrated on extending the boundary conformal field theory description of D-branes to the case of Calabi-Yau (CY) manifolds [2]–[7]. Calabi-Yau manifolds in three complex dimensions have been the subject of special attention in view of their importance of these manifolds for string compactification. (See ref. [8] for a nice summary. For earlier work that dealt with similar issues without however explicitly describing D-branes, see ref. [9].)

In the closed string case, string propagation on Calabi-Yau manifolds can be described by a variety of techniques depending on which region of the space of complex structure and Kähler moduli of the CY manifold one wishes to concentrate on. At the so-called Gepner point in the moduli space of some CY manifolds, explicit descriptions are available in terms of the tensor product of $N = 2$ conformal field theories. This point can also be described by using the Landau-Ginzburg (LG) description of these conformal field theories. The LG description provides a link between the abstract geometrical structure encoded in the CFT and a more explicit description in terms of the co-ordinates of the algebraic geometric picture of the CY manifold. The LG description can be used also for CY manifolds that may not have a corresponding Gepner construction. More generally, the LG models may be viewed as the description appropriate to a particular region in the enlarged moduli space of Calabi-Yau vacua.

For the study of D-branes one can use the corresponding extensions of these descriptions to world-sheets with boundary. In the case of the Gepner construction, one may use the boundary conformal field theory techniques due to Cardy [10], to provide an explicit construction of boundary states associated to D-branes. However to make the geometric picture of D-branes more explicit, one may, in simple cases, work with a functional integral description of such theories with an explicit lagrangian involving free bosons and free fermions. For more complicated examples of CY manifolds one would like to extend the LG description to world-sheets with boundary.

Substantial progress has been achieved in the application of the methods of boundary conformal field theory to the case of D-branes wrapped on supersymmetric cycles in the CY. Following on the work of Recknagel and Schomerus [2]that used the Gepner model construction for the description of the boundary states relevant to D-brane constructions on supersymmetric cycles on Calabi-Yau manifolds, the specific case of D-branes on the quintic Calabi-Yau manifold was studied in detail in the work of Brunner et al. [4]. Among other results, a particularly important one (and relevant to the results of this paper as we shall explain below) was their use of the identification of the Witten index in the open string sector between two boundaries with the intersection matrix between the corresponding D-branes to study systematically some properties of D-branes with both A-type and B-type boundary conditions (in the notation of Ooguri et al. [1]). Subsequent papers have utilised these techniques to particularly study B-type boundary states in other Calabi-Yau manifolds [6].

Despite this impressive progress, several important puzzles and open questions remain. It would take us too far afield to list these but there are two that are the underlying theme of the present paper. What precisely is the geometric interpretation of the large number of D-brane like boundary states that are to be found in the boundary CFTs arising from Gepner type constructions? Secondly, are there more general geometric constructions that may or may not be realised in the boundary conformal field theory approach? While the answer to the second question is generally yes, we still need to explicitly investigate such constructions. We note that these questions need to be clarified further separately in the case of A-type and B-type boundary conditions. For A-type boundary conditions one may appeal to the *modified geometric hypothesis* of ref. [4, 8]. According to this hypothesis we may expect that the masses and charges for branes with A-type boundary conditions conditions computed in the "large volume" limit continue to hold in the "small volume" limit also. Thus the A-type boundary conditions can in principle be computed in a suitable description that keeps explicit track of the geometry associated with the corresponding Gepner construction (modulo some caveats that we shall discuss later). In the case of B-type branes this is not expected to be true. Following the method developed by Brunner et al. the data at the Gepner point have to be monodromy transformed to the large volume limit (by using the monodromy transformations computed in the mirror CY) to obtain the corresponding interpretation of these branes. In the case of B-type boundary conditions the corresponding charges in the large volume descriptions have been obtained of all the boundary states obtained in several examples including the quintic. However a full geometric or physical understanding is still lacking, particularly with regard to the description at the Gepner point. In the case of the A-type boundary conditions only one class of boundary states have been tentatively identified with the corresponding geometric construction.

In this paper, as a first step in trying to answer these questions, we will investigate in detail general classes of A-type boundary conditions from a more geometric viewpoint. This will lead us to not only investigate boundary conditions related to the Recknagel-Schomerus construction but also more general constructions that clearly go beyond the Recknagel-Schomerus class.

In an earlier paper [5], the correspondence between boundary states in boundary CFT and boundary conditions in LG models was studied. This correspondence was explicitly illustrated in the case of the supersymmetric one-cycles of the two-

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torus, using the common discrete symmetries of the boundary conformal field theory and the boundary LG theory. A general class of linear boundary conditions in the LG models was also described. These are relevant to both the case of D-branes wrapped on the middle-dimensional cycles of a CY as well as the case of even-dimensional D-branes wrapped on holomorphic sub-manifolds of a CY. However an explicit identification of the boundary states of the CFT with those from the LG theory was still lacking.

In this paper we will begin by considering linear A-type boundary conditions in LG models, of the form discussed in our earlier paper. By explicitly performing the open-string Witten index calculation in the LG model and comparing it to the boundary conformal field theory calculation we will definitively show the equivalence of this class of linear boundary conditions with the $L = \lfloor k/2 \rfloor$ class of boundary states in the boundary CFT.

We will then turn to more general, generically non linear, A-type boundary conditions and show the consistency conditions that are required to ensure that these describe supersymmetric middle-dimensional cycles in Calabi-Yau manifolds. It is well known that in the case of $N = 2$ world-sheet supersymmetry several interesting features of the conformal limit are seen even when the theory is perturbed away from this limit. Hence models with $N = 2$ world-sheet supersymmetry even away from the conformal limit are of interest. In our discussion therefore we will consider A-type boundary conditions in Landau-Ginzburg descriptions of minimal models, both without and with perturbations by relevant operators. We extend this discussion to more general cases.

A summary of the main results of the paper is as follows:

1. We compute the open-string Witten index in the LG model and provide evidence that the linear class of boundary conditions in the minimal model correspond to the $L = \lfloor k/2 \rfloor$ boundary states in the minimal model.

2. We show that A-type $N = 2$ supersymmetry is preserved if the submanifold is *lagrangian*. The complete set of boundary conditions associated with this lagrangian submanifold are presented. The analysis is used to provide a microscopic description of the special lagrangian submanifolds in $\mathbb{C}^n$ due to Harvey and Lawson [11].

3. For the cases with a superpotential W (which describe hypersurfaces in $\mathbb{P}^n$), we show that one needs to have the boundary conditions have vanishing Poisson bracket with $(W - \overline{W})$. Thus, these submanifolds are necessarily pre-images of *straight lines* in the W-plane.

4. For a single minimal model, we find non-linear boundary conditions by perturbations of the quasi-homogeneous potential by relevant operators. The

boundary conditions correspond to straight lines in the W-plane passing through the minima of the perturbed potential.

5. For the case of the quintic CY threefold, we use these methods to provide an explicit microscopic description of the $\mathbb{T}^3$ special lagrangian sub-manifold in the infinite complex structure limit.

Observations closely related to items 3 and 4 above from a slightly different viewpoint[1] appear in the work of Hori, Iqbal and Vafa [12]. Some of these results have been reported in a recent paper by Hori and Vafa which appeared while this manuscript was under preparation [14].)

We will now comment on the applications as well as limitations of the LG approach pursued in this paper. As in the case of closed strings the LG model is well suited to understanding aspects involving chiral and anti-chiral primaries. In the closed-string channel, one can hope to obtain the Cardy coefficients of A-type boundary states for Ishibashi states associated with (c, c) and (a, a) primaries. In the open-string channel, for B-type boundary conditions, the spectrum of marginal/relevant boundary perturbations can also be identified as well as the boundary superpotential may be calculable. For Calabi-Yau manifolds, quantities that are independent of Kähler moduli can be conveniently studied in the LG model.

The organisation of the paper is as follows: in section 2, we discuss the case of LG models with boundary and list a linear class of boundary conditions obtained in [5]. In section 3, we review the construction of A-type boundary states in a single minimal model using Cardy's prescription. In section 4, we carry out the open-string Witten index computation and provide a map from the boundary conditions in the LG model to a class of boundary states in the corresponding CFT. This is illustrated for the case of a single minimal model and for the Gepner model associated with the quintic. In section 5, we consider general boundary conditions consistent with A-type $N = 2$ worldsheet supersymmetry. Using this microscopic description, we obtain conditions under which the boundary conditions describe a supersymmetric cycle (special lagrangian). We apply the methods to some simple examples. We conclude in section 6 with some remarks.

2. Landau-Ginzburg theories with boundary

2.1 Notation and conventions

We work in $N = 2$ superspace with coordinates x^m, θ^α, $\overline{\theta}^{\dot\alpha}$ $(m = 0, 1,\ \alpha, \dot\alpha = +, -)$. Left movers are specified by the index $-$ and right movers by the index $+$. The worldsheet has lorentzian signature (metric $=$ Diag$(-1, +1)$) and has a boundary at $x^1 = 0$ and is topologically a half-plane.

[1]These results have been reported by C. Vafa in a recent conference talk [13]. We thank C. Vafa for bringing this to our attention.

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The lagrangian for a $N=2$ supersymmetric Landau-Ginzburg theory is constructed from chiral superfields $(y^m = x^m + i\theta^\alpha \sigma^m_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha})$

$$\Phi(x,\theta) = \phi(y) + \sqrt{2}\theta^\alpha\psi_\alpha(y) + \theta^\alpha\theta_\alpha F(y)$$

and anti-chiral superfields. The lagrangian for n chiral superfields Φ_i is given by

$$S = \int d^2x d^4\theta K(\Phi,\overline{\Phi}) - \int d^2x d\theta^+ d\theta^- W(\Phi) - \int d^2x d\overline{\theta}^+ d\overline{\theta}^- \overline{W}(\overline{\Phi}) \,, \tag{2.1}$$

where K is the Kähler potential and W is the holomorphic superpotential. We will choose the Kähler potential to be $K = \sum_i \overline{\Phi}_i\Phi_i$. In the conformal case, the superpotential is taken to be quasi-homogeneous: $W(\lambda^{n_i}\Phi_i) = \lambda^d W(\Phi_i)$, where n_i are some integers which are related to the charges of the superfields Φ_i. The lagrangian takes the following form after the auxiliary fields F_i are eliminated[2]

$$\begin{aligned} S = \int d^2x \Bigg(&-\partial_m\overline{\phi}_i\partial^m\phi_i + i\overline{\psi}_{-i}\left(\overleftrightarrow{\partial_0} + \overleftrightarrow{\partial_1}\right)\psi_{-i} + i\overline{\psi}_{+i}\left(\overleftrightarrow{\partial_0} - \overleftrightarrow{\partial_1}\right)\psi_{+i} - \\ &- \left|\frac{\partial W}{\partial\phi_i}\right|^2 - \frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\psi_{-i}\psi_{+j} - \frac{\partial^2\overline{W}}{\partial\overline{\phi}_i\partial\overline{\phi}_j}\overline{\psi}_{+i}\overline{\psi}_{-j}\Bigg) \,, \end{aligned} \tag{2.2}$$

where $A \overleftrightarrow{\partial_i} B \equiv \frac{1}{2}[A(\partial_i B) - (\partial_i A)B]$.

The lagrangian is invariant under the supersymmetry transformations parametrised by ϵ_α and $\overline{\epsilon}_{\dot\alpha}$. The transformations of the fields are given by

$$\begin{aligned} \delta\phi_i &= \sqrt{2}(-\epsilon_-\psi_{+i} + \epsilon_+\psi_{-i}) \,, \\ \delta\psi_{+i} &= i\sqrt{2}(\partial_0 + \partial_1)\phi_i\overline{\epsilon}_- + \sqrt{2}\epsilon_+\frac{\partial\overline{W}}{\partial\overline{\phi}_i} \,, \\ \delta\psi_{-i} &= -i\sqrt{2}(\partial_0 - \partial_1)\phi_i\overline{\epsilon}_+ + \sqrt{2}\epsilon_-\frac{\partial\overline{W}}{\partial\overline{\phi}_i} \,. \end{aligned} \tag{2.3}$$

We will be interested in considering the case when boundary conditions preserve part of the supersymmetry. Further, the boundary conditions should cancel the ordinary variations of the action modulo the bulk equations of motion. The ordinary variation of the action gives rise to the following boundary terms

$$\begin{aligned} \delta_{\mathrm{ord}}S = \int dx^0 \Bigg(&\left[\left(\partial_1\overline{\phi}_i\right)\delta\phi + \delta\overline{\phi}_i(\partial_1\phi_i)\right]\Big|_{x^1=0} + \\ &+ \frac{i}{2}\left[\delta\overline{\psi}_{-i}\psi_{-i} - \overline{\psi}_{-i}\delta\psi_{-i} - \delta\overline{\psi}_{+i}\psi_{+i} + \overline{\psi}_{+i}\delta\psi_{+i}\right]\Big|_{x^1=0}\Bigg) \,. \end{aligned} \tag{2.4}$$

There are two inequivalent sets of boundary conditions which preserve different linear combinations of the left and right $N=2$ supersymmetries [1, 15].

[2] In addition, we have symmetrised the action of the derivatives occuring in the Kinetic energy term for the fermions as is done when one is considering worldsheets with boundary.

A-type boundary conditions. These are boundary conditions such that the unbroken $N = 2$ supersymmetry is generated by

$$\epsilon_+ = \eta\, \bar{\epsilon}_- \,, \tag{2.5}$$

and the complex conjugate equation and $\eta = \pm 1$ corresponds to the choice of spin-structure on the worldsheet.

B-type boundary conditions. These are boundary conditions such that the unbroken $N = 2$ supersymmetry is generated by

$$\epsilon_+ = \eta\, \epsilon_- \,, \tag{2.6}$$

and the complex conjugate equation and $\eta = \pm 1$ corresponds to the choice of spin-structure on the worldsheet. The two boundary conditions are related by the mirror automorphism of the $N = 2$ supersymmetry algebra under which the left-moving U(1) current changes sign.

2.2 A-type boundary conditions

Under A-type boundary conditions, the unbroken $N = 2$ supersymmetry is given by the condition

$$\epsilon_+ = \eta\, \bar{\epsilon}_- \,, \tag{2.7}$$

where $\eta = \pm 1$. In an earlier paper, it was shown that the following conditions[3] preserve $N = 2$ supersymmetry and that the boundary terms in ordinary variations (eq. (2.4)) of the lagrangian vanish.

$$\begin{aligned}
\left(\psi_{+i} - A_{ij}\eta\, \overline{\psi}_{-j}\right)\Big|_{x^1=0} &= 0\,, &\qquad \partial_1\left(\phi_i + A_{ij}\overline{\phi}_j\right)\Big|_{x^1=0} &= 0\,,\\
\partial_0\left(\phi_i - A_{ij}\overline{\phi}_j\right)\Big|_{x^1=0} &= 0\,, &\qquad \left(A_{ij}\frac{\partial W}{\partial \phi_j} - \frac{\partial \overline{W}}{\partial \overline{\phi}_i}\right)\Bigg|_{x^1=0} &= 0\,,
\end{aligned} \tag{2.8}$$

where A is a symmetric matrix satisfying $AA^\dagger = 1$.

For the k-th minimal model, the LG description has a superpotential given by $W = \phi^{k+2}/(k+2)$, the condition involving the superpotential becomes

$$A^{k+2} = 1\,, \tag{2.9}$$

which is a condition on the parameter A appearing in the boundary condition. Thus, A can be any $(k+2)$-th root of unity. Hence there are $(k+2)$ different boundary conditions which are consistent with $N = 2$ supersymmetry.

Under the action of the generator g of the group $\mathbb{Z}_{k+2}$, one can easily check that $A \to A\exp(4\pi i/(k+2))$. Suppose we choose to label the different $(k+2)$ roots of unity by

$$A_m = \exp(2\pi m/k+2)\,.$$

[3]The boundary conditions have been adapted to the notation used in this paper.

Then under the action of g, $A_m \to A_{m+2}$. This suggests that the m label here can be associated with the M labels of the boundary states constructed in the corresponding minimal model. For odd k, the allowed values of A form a $(k+2)$-dimensional orbit while for even $k = 2n$, one obtains two $(n+1)$-dimensional orbits of the $\mathbb{Z}_{n+1}$ subgroup of the $\mathbb{Z}_{k+2}$.

2.3 B-type boundary conditions

Under B-type boundary conditions, the unbroken $N = 2$ supersymmetry is given by the condition

$$\epsilon_+ = \eta\, \epsilon_- \,, \tag{2.10}$$

where $\eta = \pm 1$. The following linear boundary conditions were constructed in the LG model [5]

$$\begin{aligned} \left(\psi_{+i} + \eta B_i{}^j \psi_{-j}\right)\Big|_{x=0} &= 0\,, & \partial_1 \left(\phi_i + B_i{}^j \phi_j\right)\Big|_{x=0} &= 0\,, \\ \partial_0 \left(\phi_i - B_i{}^j \phi_j\right)\Big|_{x=0} &= 0\,, & \left(\frac{\partial W}{\partial \phi_i} + B_i^{*j} \frac{\partial W}{\partial \phi_j}\right)\Bigg|_{x=0} &= 0\,, \end{aligned} \tag{2.11}$$

where the boundary condition is specified by a hermitean matrix B which satisfies $B^2 = 1$. Since B squares to one, its eigenvalues are ± 1. An eigenvector of B with eigenvalue of $+1$ corresponds to a Neumann boundary condition and -1 corresponds to a Dirichlet boundary condition. Associated with every eigenvector with eigenvalue $+1$, there is a non-trivial condition involving the superpotential which is given by the last of the above boundary conditions.

For a LG model with a single chiral superfield such as the minimal model, the consistency condition involving the superpotential does not permit the imposition of a Neumann boundary condition on the scalar field. Thus, one can only impose Dirichlet boundary conditions on the scalar. We will not consider B-type boundary states further in this paper.

3. Boundary states in the $N = 2$ minimal models

3.1 Notation and conventions

The k-th $N = 2$ minimal models has central charge $c = 3k/(k+2)$. The primary fields of the model are labelled by three integers (l, m, s) with

$$\begin{aligned} l &= 0, \ldots, k\,, \\ m &= -(k+1), -k, \ldots, (k+2) \bmod (2k+4)\,, \\ s &= -1, 0, 1, 2 \bmod 4\,, \end{aligned}$$

subject to the constraint that $l+m+s$ is even. In addition there is a field identification given by

$$(l, m, s) \sim (k - l, m + k + 2, s + 2)\,.$$

Even s refers to the NS sector and odd s refers to the R sector fields.

A complete set of labels for the minimal model (using the field identification mentioned above) are given by

$$l = 0, \ldots, \lfloor k/2 \rfloor\,, \qquad m = -(k+1), -k, \ldots, (k+2) \qquad \text{and} \qquad s = -1, 0, 1, 2\,,$$

where $\lfloor k/2 \rfloor$ is the largest integer less than or equal to $k/2$ and $(l + m + s)$ is even.

Another equivalent set of labels is given by

$$l = 0, \ldots, k\,, \qquad m = -(k+1), -k, \ldots, (k+2) \qquad \text{and} \qquad s = 0, 1\,,$$

where again we have the condition that $(l + m + s)$ must be even. The dimension h and U(1) charge q of the fields are given by

$$h_{l,m,s} = \frac{l(l+2) - m^2}{4(k+2)} + \frac{s^2}{8}\,, \qquad q_{l,m,s} = \frac{m}{k+2} - \frac{s}{2} \bmod 2\,. \tag{3.1}$$

The k-th minimal model has a $\mathbb{Z}_{k+2} \times \mathbb{Z}_2$ discrete symmetry. The action of the discrete symmetry on the fields is given by

$$\begin{aligned} g \cdot \Phi_{l,m,s} &= e^{\frac{2\pi i m}{k+2}}\ \Phi_{l,m,s}\,, \\ h \cdot \Phi_{l,m,s} &= (-)^s\ \Phi_{l,m,s}\,, \end{aligned} \tag{3.2}$$

where g and h generate the $\mathbb{Z}_{k+2}$ and $\mathbb{Z}_2$, respectively. We will be interested in the action of $\exp(i\pi J_0)$ on a bulk state:

$$\exp(i\pi J_0)|l, m, s\rangle = \exp\left(i\pi\left[\frac{m}{k+2} - \frac{s}{2}\right]\right)|l, m, s\rangle\,. \tag{3.3}$$

Note that this is not necessarily equal to $(-)^{F_L}$ when one is considering a single minimal model (as opposed to a Gepner construction involving integer U(1) charges). However, one can see that product $(-)^{F_L}\exp(i\pi J_0)$ commutes with all generators of the $N = 2$ supersymmetry algebra. In the minimal model, this product on general grounds should be given by $f(g, h)$, where f is some function of the the discrete symmetries which commute with $N = 2$ supersymmetry [16]. Thus, we will also be interested in defining the action of $(-)^{F_L}$ on the states $|l, m, s\rangle$. We will require that $(-)^{F_L}$ gives ± 1 acting on the NS sector states. For odd k, we find

$$(-)^{F_L}|l, m, s\rangle = \exp\left(i\pi\left[m + \frac{s}{2}\right]\right)|l, m, s\rangle\,. \tag{3.4}$$

The above assignment is consistent with the identification $|l, m, s\rangle \sim |k - l, m + k + 2, s + 2\rangle$ of states. Thus, for odd k, $f(g, h) = g^{(k+3)/2}$.

For a given representation p of the $N=2$ algebra, the character is defined as

$$\chi_p\,(q,z,u) = e^{-2i\pi u}\ \mathrm{Tr}_p\left[e^{2i\pi z J_0}\ q^{(L_0-c/24)}\right], \tag{3.5}$$

where $q=\exp(2i\pi\tau)$ and u is an arbitrary phase. The trace runs over the representation denoted by p. The characters of the $N=2$ minimal models are defined in terms of the Jacobi theta functions $\theta_{n,m}(\tau,z,u)$ and characters of a related parafermionic theory $C^l_m(\tau)$ as:

$$\chi^{(s)}_{l,m}\,(q,z,u) = \sum_{j \bmod k} C^l_{m+4j-s}(\tau)\ \theta_{2m+(4j-s)(k+2),2k(k+2)}\,(\tau,2kz,u)\,. \tag{3.6}$$

The characters $\chi^{(s)}_{l,m}$ have the property that they are invariant under $s\to s+4$ and $m\to m+2(k+2)$ and are zero if $(l+m+s)$ is odd. By using the properties of the theta functions, the modular transformation of the minimal model characters is found to be

$$\chi^{(s)}_{l,m}\,(\hat{q},0,0) = C\sum_{l',m',s'} \sin(l,l')_k \exp\left(\frac{i\pi m m'}{k+2}\right)\exp\left(-\frac{i\pi s s'}{2}\right)\chi^{(s')}_{l',m'}(q,0,0)\,, \tag{3.7}$$

where $\hat{q}=\exp(-2i\pi/\tau)$; $(l,l')_k \equiv \left(\frac{\pi(l+1)(l'+1)}{k+2}\right)$ and $C=1/\sqrt{2}(k+2)$.

3.2 A-type boundary states in the $N=2$ minimal models

We will consider A-series which has a diagonal partition function. For A-type boundary conditions, there are Ishibashi states [17] for each possible value of (l,m,s). We will label these states by $|l,m,s\rangle\rangle$. Using Cardy's prescription [10], we can construct the boundary states

$$|L,M,S\rangle = \sqrt{C}\sum_{l,m,s}\frac{\sin(L,l)_k}{\left[\sin(l,0)_k\right]^{-1/2}}\exp\left(\frac{i\pi M m}{k+2}\right)\exp\left(-\frac{i\pi S s}{2}\right)|l,m,s\rangle\rangle\,, \tag{3.8}$$

where we have used upper case letters to represent the boundary state and lower case letters for the Ishibashi states. One can check that the boundary states $|L,M,S\rangle$ and $|L,M,S+2\rangle$ differ only in the sign occurring in front of the RR-sector (i.e. odd s) Ishibashi states. Thus, it suffices to study only the $S=0,1$ states.

The field identification $(l,m,s)\sim(k-l,m+k+2,s+2)$ in the bulk minimal model extends to the boundary states as $|L,M,S\rangle \sim |k-L,M+k+2,S+2\rangle$. The annulus amplitude $\mathcal{A}_{L,M,S}(q)$ (with modular parameter q) which is given by the modular transform of the cylinder amplitude $\langle 0,0,0|\hat{q}^{L_0-c/24}|L,M,S\rangle$, is given by

$$\mathcal{A}_{L,M,S}(q) = \chi^{(S)}_{L,M}(q)\,. \tag{3.9}$$

Note that this vanishes when $(L+M+S)$ is odd. Thus, we impose the additional condition that $(L+M+S)$ be even.

The full set of boundary states that we obtain are specified by the following values of (L, M, S):

$$L = 0, \ldots, \lfloor k/2 \rfloor\,, \qquad M = -(k+1), -k, \ldots, (k+2) \qquad \text{and} \qquad S = 0, 2\,.$$

In this labelling convention, we will sometimes loosely refer to the $S = 2$ state as an antibrane (in analogy with the situation in the full Gepner construction) since the $S = 2$ boundary state differs from the $S = 0$ state (for identical values of L, M) by an overall sign in front of the RR Ishibashi states. This set of labels takes care of the identification of boundary states mentioned earlier except for the case when k is even and $L = k/2$. For this case, the antibrane corresponding to the boundary state $|k/2, M, 0\rangle$ is $|k/2, M + k + 2, 0\rangle$.

Under the discrete symmetries of the minimal model $\mathbb{Z}_{k+2} \times \mathbb{Z}_2$, the boundary states transform as

$$\begin{aligned} g \cdot |L, M, S\rangle &= |L, M + 2, S\rangle\,, \\ h \cdot |L, M, S\rangle &= |L, M, S + 2\rangle\,. \end{aligned} \tag{3.10}$$

Thus all A-type boundary states can be classified into orbits of the discrete symmetry. When k is odd, there are $\lfloor k/2 \rfloor = (k-1)/2$ orbits of length $(k+2)$ after taking into account the identification of the boundary states mentioned earlier. For even k, when $l < k/2$, then the states are in orbits of length $(k+2)$. However, when $l = k/2$, since $l = k - l$, the orbit length is shorter and equals $(k+2)/2$ (provided one ignores the distinction between the $S = 0$ and $S = 2$ states).

The characters of the full $N = 2$ supersymmetry algebra are given by the combination $(\chi^{(s)}_{lm} + \chi^{(s+2)}_{lm})$. It is thus of interest to construct boundary states corresponding to these characters. In this regard consider

$$|L, M, \pm\rangle \equiv \frac{1}{\sqrt{2}}(|L, M, S\rangle \pm |L, M, S + 2\rangle)\,. \tag{3.11}$$

From the earlier discussion, it is clear that the states $|L, M, +\rangle$ involve Ishibashi states from the NSNS sector and $|L, M, -\rangle$ involve Ishibashi states from the RR sector. These states also are more natural in the construction of boundary states in the Gepner model since the tensor product of boundary states $\prod_i |L_i, M_i, +\rangle$ incorporates the condition that NSNS states of each sub-theory (labelled by i) are tensored to each other and the tensor product of boundary states $\prod_i |L_i, M_i, -\rangle$ works similarly for the RR states. The annulus amplitude $\mathcal{A}_{L,M,\pm}(q)$ (with modular parameter q) which is given by the modular transform of the cylinder amplitude $\langle 0, 0, +|\hat{q}^{L_0 - c/24}|L, M, \pm\rangle$, is given by

$$\mathcal{A}_{L,M,\pm}(q) = \chi^{(S)}_{L,M}(q) \pm \chi^{(S+2)}_{L,M}(q)\,, \tag{3.12}$$

where $S = L + M \bmod 2$. Under the discrete symmetries of the minimal model $\mathbb{Z}_{k+2} \times \mathbb{Z}_2$, the boundary states transform as

$$\begin{aligned} g \cdot |L, M, \pm\rangle &= |L, M+2, \pm\rangle\,, \\ h \cdot |L, M, \pm\rangle &= \pm|L, M, \pm\rangle\,. \end{aligned} \tag{3.13}$$

As before, all states except the case when k is even and $L = k/2$, the states can be arranged in $\mathbb{Z}_{k+2}$ orbits. However, when k is even and $L = k/2 = n$, the orbit length is shorter. One has

$$g^{n+1} \cdot |L, M, \pm\rangle = \pm|L, M, \pm\rangle\,.$$

Thus, they have orbit length $(n+1) = (k+2)/2$.

4. Computing the open-string Witten index

We have so far constructed A-type boundary conditions in the Landau-Ginzburg model and constructed A-type boundary states in the corresponding minimal model. However, since the number of boundary conditions is smaller than the number of states, we would like to identify to which boundary states to which they correspond. This is not easy to do even in simple cases such as the Ising model with boundary. A useful tool in this regard is to classify boundary conditions and boundary states in terms of discrete symmetries such as the $\mathbb{Z}_2$ in the Ising model. In the present problem, as we have already seen, there is a $\mathbb{Z}_{k+2}$ discrete group which organises the boundary conditions and boundary states into orbits. It turns out that this alone is sufficient to provide an identification for the even k minimal model. The LG boundary conditions form two orbits of the $\mathbb{Z}_{(k+2)/2}$ subgroup of $\mathbb{Z}_{k+2}$. This uniquely identifies them with the $L = k/2$ boundary states.

This is not the case for odd k where the LG boundary conditions form a single $\mathbb{Z}_{k+2}$ orbit which is true of all boundary states in the corresponding minimal model. In order to make the identification, we will use a open-string Witten index computation (due to Douglas and Fiol [18]). In the context of Calabi-Yau threefolds, this index computes the intersection matrix between three cycles. We will compute the index in both the LG as well as boundary CFT and show that the LG boundary conditions correspond to the $L = \lfloor k/2 \rfloor$ boundary states.

Let $|B\rangle$ and $|B'\rangle$ be two boundary states. The Witten index is defined as [18]

$$\widetilde{\mathcal{I}_{BB'}} = {}_{\rm RR}\langle B'|\; (-)^{F_L}\; \tilde{q}^{(L_0 - c/24)}\; |B\rangle_{\rm RR}\,, \tag{4.1}$$

where $|B\rangle_{\rm RR}$ refers to the RR part of the boundary state. In the open-string channel, this counts the number of *Ramond ground states* of the hamiltonian $H_{BB'}$:

$$\widetilde{\mathcal{I}_{BB'}} = \mathrm{Tr}_{BB'}\left[(-)^F q^{(L_0 - c/24)}\right]. \tag{4.2}$$

As discussed by Witten [16], the operator $(-)^{F_L}$ can be replaced by $\exp(i\pi J_0^L)$ where J_0^L is the zero-mode of the left-moving U(1) current. Thus, we will be computing the following object:

$$\mathcal{I}_{BB'} = {}_{\rm RR}\langle B'|\ e^{i\pi J_0^L}\ \hat{q}^{(L_0-c/24)}\ |B\rangle_{\rm RR}\,. \tag{4.3}$$

In the open-string channel, this will be given by

$$\mathcal{I}_{BB'} = \mathrm{Tr}_{BB'}\left[e^{i\pi J_0} q^{(L_0-c/24)}\right], \tag{4.4}$$

where J_0 is the charge associated with the unbroken U(1).

For the level k (k odd) minimal model, one can study the action of $(-)^{F_L}$ and $\exp(i\pi J_0^L)$ on the boundary states. For A-type boundary states, we can see that

$$\begin{aligned}\exp(i\pi J_0^L)|L,M,S\rangle &= |L,M+1,S+1\rangle\,,\\ (-)^{F_L}|L,M,S\rangle &= |L,M+k+2,S-1\rangle\,.\end{aligned} \tag{4.5}$$

Thus, we see that for A-type boundary states (and odd k)

$$\exp(i\pi J_0^L) = g^{\frac{k+3}{2}} h (-)^{F_L}\,. \tag{4.6}$$

4.1 Boundary minimal model calculation

We will now compute the following in the boundary minimal model

$$\mathcal{I}_{L,M,0;L',M',0} \equiv {}_{\rm RR}\langle L',M',0|\ \exp(i\pi J_0)\,\hat{q}^{(L_0-c/24)}\,|L,M,0\rangle_{\rm RR}\,. \tag{4.7}$$

This calculation is identical to the one in the appendix of ref. [4] tailored to the case of a single minimal model. We reproduce it here for completeness. Using the expression for the boundary states constructed using Cardy's prescription, we get

$$\begin{aligned}\mathcal{I}_{L,M,0;L',M',0} = C\sum_{l,m,s}^{R} \frac{\sin(L,l)_k \sin(L',l)_k}{\sin(l,0)_k} \exp\left(\frac{i\pi(M-M'+1)m}{k+2}\right)\times\\ \times\, e^{-i\pi s/2}\chi^s_{lm}(\hat{q})\,,\end{aligned} \tag{4.8}$$

where the R in the summation refers to the restriction to the Ramond sector (i.e, $s=\pm1$). On transforming to the open-string channel by an S-transformation, one obtains

$$\begin{aligned}\mathcal{I}_{L,M,0;L',M',0} &= C^2\sum_{l,m,s}^{R}\sum_{l',m',s'} \frac{\sin(L,l)_k \sin(L',l)_k \sin(l.l')_k}{\sin(l,0)_k} e^{\left(\frac{i\pi\mu m}{k+2}\right)} e^{-i\pi s(1+s')/2}\chi^{s'}_{l'm'}(q)\\ &= -2C^2\sum_{l,m}^{R}\sum_{l',m'}^{R} \frac{\sin(L,l)_k \sin(L',l)_k \sin(l.l')_k}{\sin(l,0)_k} e^{\left(\frac{i\pi\mu m}{k+2}\right)} I^{m'}_{l'}(q)\,,\end{aligned} \tag{4.9}$$

where $\mu \equiv M - M' + m' + 1$ and

$$I^m_l(q) \equiv \chi^{(1)}_{l,m}(q) - \chi^{(-1)}_{l,m}(q) = \delta_{m,l+1} - \delta_{m,-l-1}\,.$$

(See ref. [19] for the above relation.) In the above, we have carried out the s and s' summations and hence the restriction R now implies that $(l+m)$ and $(l'+m')$ must be odd. On carrying out the summation over m, we get

$$\mathcal{I}_{L,M,0;L',M',0} = -2C^2(k+2)\sum_{l=0}^{k}\sum_{l',m'}^{R} \frac{\sin(L,l)_k \sin(L',l)_k \sin(l.l')_k}{\sin(l,0)_k} \times$$
$$\times\, \delta_{\mu}^{(k+2)}(-)^{\frac{\mu(l+1)}{(k+2)}} I_{l'}^{m'}(q)\,, \tag{4.10}$$

where $\delta_{\mu,0}^{(k+2)}$ is the periodic delta function of period $(k+2)$, i.e. it is non vanishing for $\mu = 0 \bmod (k+2)$. We can now carry out the summation over l. We then obtain

$$\mathcal{I}_{L,M,0;L',M',0} = -C^2(k+2)^2 \sum_{l',m'}^{R} N_{LL'}^{l'}\,\delta_{M-M'+m'+1}^{(2k+4)} I_{l'}^{m'}(q)\,, \tag{4.11}$$

where $N_{LL'}^{l'}$ is the SU(2) level k fusion coefficient and $\delta_M^{(2k+4)}$ is the periodic delta function with period $(2k+4)$. On carrying out the summations over l' and m' we obtain (after substituting for C)

$$\mathcal{I}_{L,M,0;L',M',0} = N_{L,L'}^{M-M'}\,, \tag{4.12}$$

where we have continued the top index M of the SU(2) fusion coefficient $N_{L,L'}^{M}$ to values mod$(2k+4)$ following the work of Brunner et al. [4]. The continuation is given by

$$N_{L,L'}^{-l-2} = -N_{L,L'}^{l} \qquad \text{and} \qquad N_{L,L'}^{-1} = N_{L,L'}^{k+1} = 0\,,$$

where $l = 0, \ldots, k$. Thus the intersection number is given by the appropriate fusion coefficient. Following ref. [4], we can write the fusion coefficient $N_{L,L'}^{M}$ as a matrix in the index M. This matrix can be represented as a polynomial in g, the generator of $\mathbb{Z}_{k+2}$. Using this presentation of the fusion coefficient, the Witten index for odd k and $L = L' = (k-1)/2$ is given by

$$\mathcal{I}^{\mathrm{mm}} = \left(1 + g + \cdots + g^{(k-1)/2} - g^{-1} - g^{-2} - \cdots - g^{(-k-1)/2}\right), \tag{4.13}$$

where we use the superscript mm to denote that this is a minimal model computation.

4.2 Boundary Landau-Ginzburg calculation

We will now compute the Witten index in the LG model. The worldsheet is assumed to have the topology of an annulus (of width π). We will impose A-type boundary conditions at the two ends of the strip i.e. at $x^1 = 0$ and $x^1 = \pi$. We Wick rotate the time coordinate to euclidean space and make it periodic. At $x^1 = \pi$. we impose the condition

$$\partial_1 \operatorname{Re}\phi = \operatorname{Im}\phi = 0\,, \qquad \psi_+ = \overline{\psi}_-\,. \tag{4.14}$$

This corresponds to the choice $A = 1$ in the notation of the earlier section. At $x^1 = 0$, we impose

$$\partial_1 \,\mathrm{Re}\left(\exp\left(-\frac{i\pi m}{k+2}\right)\phi\right) = \mathrm{Im}\left(\exp\left(-\frac{i\pi m}{k+2}\right)\phi\right) = 0\,,$$
$$\psi_+ = \exp\left(\frac{2i\pi m}{k+2}\right)\overline{\psi}_-\,. \tag{4.15}$$

This corresponds to the choice $A_m = \exp(\frac{2i\pi m}{k+2})$.

We will use the doubling trick to convert the annulus into a torus. The doubling for fermions is done by introducing the extended fermion $\Psi(x^1, x^0)$.

$$\Psi(x^1,t) = \begin{cases} \psi_-(x^1,t) & \text{for } 0 \le \sigma \le \pi \\ \overline{\psi}_+(2\pi - x^1, t) & \text{for } \pi \le x^1 \le 2\pi\,. \end{cases} \tag{4.16}$$

This automatically imposes the condition $\psi_- = \overline{\psi}_+$ at the boundary at $x^1 = \pi$. The boundary condition at $x^1 = 0$ becomes the periodicity on the extended fermion Ψ. The bosonic fields can also be doubled in a similar fashion.

The counting of the Ramond ground states for the above situation can now be seen to be identical to the counting of Ramond ground states in the twisted sector of a certain orbifold: it is the m-th twisted sector of the orbifolding of the k-th minimal model by $\mathbb{Z}_{k+2}$. This computation has been carried out by Vafa and we quote his result [20]. For $m \neq 0$, there is precisely one ground state. This observation more or less uniquely identifies the boundary condition with the $l = \lfloor k/2 \rfloor$ boundary states. From eq. (4.13), where we have given the Witten index for the $l = \lfloor k/2 \rfloor$ boundary states: one can clearly see that there is typically one Ramond ground state in all sectors except one. For $m = 0$, i.e. for the case where one has identical boundary conditions on both ends of the annulus, one is dealing with the untwisted sector. In this sector, free field methods cannot be used. However, one has $(k+1)$ Ramond ground states. However, none of them satisfy the $J_L = -J_R$ boundary condition in the open-string channel. Thus, all Ramond ground states are projected out and the Witten index is zero. (If k were even, there is one Ramond ground state with vanishing left and right U(1) charges and hence the Witten index is one.)

In order to completely carry out the full LG computation, we need to suitably assign a fermion number to the ground state. Let us assign fermion number $(-)^m$ to the Ramond ground state[4] of the boundary condition given by A_m, for $m = 1, \ldots, k+1$. Using the conventions of Brunner et al., we can rewrite the above result as (for odd k)

$$\mathcal{I}^{\mathrm{LG}} = \left(g + \cdots + g^{(k+1)/2} - g^{(k+3)/2} - \cdots - g^{k+1}\right), \tag{4.17}$$

[4]The interchange of boundary conditions at $x^1 = 0, \pi$ can be represented by $A_m \to A_{-m}$. This choice makes the Witten index antisymmetric under the exchange.

where we use the superscript LG to indicate that the Witten index was computed in the LG model. Note that

$$\mathcal{I}^{\rm mm} = -g^{(k+1)/2}\mathcal{I}^{\rm LG}\,. \tag{4.18}$$

This difference can be understood as follows: the computation in the LG model is a Witten index computation while the minimal model computation is one where $\exp(i\pi J_0)$ replaces $(-)^F$. Equation (4.6) provides the relation between the two operations (on the boundary states). Thus one interprets the $\exp(i\pi J_0)$ to correspond to an additional time-twisting by $g^{(k+3)/2}$ in the Witten index computation done in the doubled theory. Following the method used in closed string LG orbifolds as in ref. [20], this can be seen to be be equivalent to an additional space-twisting by $g^{-(k+3)/2}$ which provides the required shift of $g^{(k+1)/2}$ in the calculation. The minus sign comes from the action of h, which maps branes to anti-branes and thus changes the sign in the Witten index computation. Thus, this identifies the LG boundary conditions with the $L = (k-1)/2$ boundary states for odd k.

4.3 Landau-Ginzburg orbifolds

The work of Greene, Vafa and Warner [21] showed the relationship between certain LG orbifolds and Gepner models. For example, the Gepner model description of the Calabi-Yau threefold, the quintic, at a special point in its moduli is given by the tensor product of five copies of $k = 3$ minimal models subject to certain projections [22]. The LG description involves five chiral superfields Φ_i with superpotential

$$W(\Phi) = \Phi_1^5 + \Phi_2^5 + \cdots + \Phi_5^5\,.$$

Further, the orbifolding corresponds to the identification $\phi_i \sim \alpha\phi_i$, for all $i = 1, \ldots, 5$ (α is a non-trivial fifth root of unity). It was argued by Greene, Vafa and Warner [21], that this condition is equivalent to the integer U(1) projection in the corresponding Gepner model.

In the boundary LG, we find the following set of A-type boundary conditions given by the matrix $A(\{m_i\}) = \mathrm{Diag}(\alpha^{m_1}, \ldots, \alpha^{m_5})$ [5]. The equivalence relation mentioned earlier implies that the two matrices given by parameters $\{m_i\}$ and $\{m_i + 2\}$ are equivalent. As before, we would like to calculate the Witten index which is equivalent to the intersection matrix for these cycles.

Before orbifolding it is clear that the intersection matrix is simply the product of the intersection matrix of the individual theories. The further orbifolding by the diagonal $\mathbb{Z}_5$(which results in the integer U(1) charge projection) is implemented as a projection (and hence time-twisting) in the closed-string channel. This will therefore show up as a sum over space-twisted sectors in the open-string channel. Hence in the computation of the Witten index in the doubled torus the final result is as follows:

$$\mathcal{I} = \sum_{\nu=0}^{4}\prod_{i=1}^{5} N_{1,1}^{m_i - m'_i - 4 + 2\nu}\,, \tag{4.19}$$

where we $\{m_i\}$ and $\{m'_i\}$ are the boundary conditions at the two ends of the annulus and $N^M_{1,1}$ is the $\mathrm{SU}(2)_3$ fusion coefficent. The end result can be written in the compact notation of ref. [4] as

$$\mathcal{I} = \prod_{i=1}^{5} \left(g_i + g_i^2 - g_i^3 - g_i^4\right) \tag{4.20}$$

subject to the condition that $g_1g_2g_3g_4g_5 = 1$. The result is as written in ref. [4] and is consistent with the $L = 1$ assignment of boundary state labels for each individual minimal model.

The LG calculation, especially the part involving summing over twisted sectors, is quite similar to the spacetime intersection matrix calculation (see section 2 of ref. [4]). In the spacetime calculation, the intersection calculation involves summing over patches which becomes different twisted sectors in the LG orbifold computation. For a single minimal model, we needed to explain the shift between the $(-)^F$ and $\exp(i\pi J_0)$ computations. However, for the LG orbifold, the two results are identical due to the condition $g_1g_2g_3g_4g_5 = 1$. Finally, the similarity with the spacetime intersection calculation suggests that the expectation from the modified geometric hypothesis that the central charges of the A-branes should be the same at different points in the Kähler moduli space is true.

The methods used here clearly apply to more general examples involving the linear boundary conditions in LG models as discussed in section 2. We will later see more general conditions where the Witten index computation is quite difficult.

5. Non-linear boundary conditions in LG models

As we have seen, the boundary LG description seems to provide fewer boundary conditions than the corresponding boundary minimal model. This situation holds for more general examples such as the one involving LG description of Calabi-Yau manifolds. The class of boundary conditions considered in [5] correspond to the linear class. We shall now try to generalise these conditions and see if we can obtain new conditions.

5.1 LG models with a single chiral superfield

We shall first consider the simplest case of an LG model involving a single chiral superfield Φ. The most general boundary condition is given by

$$F(\phi, \overline{\phi}) = 0\,, \tag{5.1}$$

where $F(\phi, \overline{\phi})$ is a real function. We will have to impose additional conditions such that A-type $N = 2$ supersymmetry $\epsilon_+ = \eta\overline{\epsilon}_-$ is preserved and all boundary terms (eq. (2.4)) which appear in the ordinary variation of the lagrangian vanish. In order to achieve this, we will first consider all new conditions generated under the unbroken A-type $N=2$ supersymmetry.

The first supersymmetric variation leads to the following condition:

$$\frac{\partial F}{\partial \phi}\ \psi_+ + \eta \frac{\partial F}{\partial \overline{\phi}} \overline{\psi}_- = 0\,. \tag{5.2}$$

The supersymmetric variation of the above equation leads to the following additional conditions:[5]

$$\begin{aligned} \left[\frac{\partial F}{\partial \phi}\ \partial_1 \phi - \frac{\partial F}{\partial \overline{\phi}} \partial_1 \overline{\phi}\right] - i \left[\frac{\partial F}{\partial \phi}\frac{\partial F}{\partial \overline{\phi}}\right]^{1/2} K \psi_- \overline{\psi}_- &= 0 \\ \left\{F, W(\phi) - \overline{W}(\overline{\phi})\right\}_{PB} &= 0\,, \end{aligned} \tag{5.3}$$

where K is the extrinsic curvature of the curve $F=0$ in the complex ϕ-plane given by

$$K = -\left[\frac{\partial F}{\partial \phi}\frac{\partial F}{\partial \overline{\phi}}\right]^{-3/2} \left[\left(\frac{\partial F}{\partial \overline{\phi}}\right)^2 \frac{\partial^2 F}{\partial \phi^2} - 2\left|\frac{\partial F}{\partial \phi}\right|^2 \frac{\partial^2 F}{\partial \phi \partial \overline{\phi}} + \left(\frac{\partial F}{\partial \phi}\right)^2 \frac{\partial^2 F}{\partial \overline{\phi}^2}\right].$$

One can check that the boundary terms in the ordinary variation vanish under these boundary conditions. The linear cases discussed in section 2 correspond to the case when $K=0$ (since the boundary curves are straight lines in the ϕ-plane) and are clearly seen to be special case of the more general boundary condition $F=0$.

The vanishing of the Poisson bracket $\{F, W(\phi) - \overline{W}(\overline{\phi})\}_{PB}$ imposes an important restriction on the possible boundary curves in the ϕ-plane. Since in a two-dimensional phase space, there can be at most one constant of motion, the only possible boundary condition is

$$F = W(\phi) - \overline{W}(\overline{\phi}) - ic\,, \tag{5.4}$$

where c is a real constant. These correspond to *straight lines* in the W-plane which are parallel to the real W axis. For the case when $W = \phi^{k+2}/(k+2)$, the pre-image of $F=0$ in the ϕ-plane will generically have $(k+2)$ components. When $c=0$, these $(k+2)$ pre-images are precisely the $(k+2)$ linear boundary conditions that we have already obtained!

We can now discuss as to how the other boundary conditions will appear in the LG model. In this regard, we would like to make the following observations: (i) The superpotential W has $(k+1)$ degenerate minima at $\phi=0$. (ii) We will require that all the curves $F=0$ should pass through the minima which fixes the constant $c=0$. (iii) The minima can be made non degenerate by deforming the potential. A possible deformation is to add $-\lambda\phi$ to the superpotential. This leads

[5] In the following, we define the Poisson bracket using the symplectic structure naturally associated with the Kähler potential of the LG model. Explicitly, when the Kähler potential is $\phi\overline{\phi}$, $\{A,B\}_{PB} = 2i(\partial A \overline{\partial} B - \partial B \overline{\partial} A)$ where $\partial \equiv \partial/\partial\phi$ and $\overline{\partial} \equiv \partial/\partial\overline{\phi}$.

to non-degenerate minima located at the $(k+1)$ roots of λ. By a suitable rescaling, we can set $\lambda = 1$. We will require that the only allowed values of the constant c are such that $F = 0$ passes through one of the minima.[6]

Thus, we propose that the $(k+1)$ boundary states (labelled by $L = 0, \ldots, k$) correspond to the $(k+1)$ boundary conditions in the LG model given by the pre-images of the the straight lines in the W-plane:

$$F_l(\phi, \overline{\phi}) = W(\phi) - \overline{W}(\overline{\phi}) - ic_l = 0\,,$$

where $c_l = 2\,\mathrm{Im}\,W(\phi_l)$, where ϕ_l are the $(k+1)$ minima of the bosonic potential. However, obtaining the map between the L label associated with the boundary state and the l label associated with the boundary condition will require the calculation of the Witten index. Since we are as yet unable to compute the Witten index in these non-linear situations, the identification cannot be made more precise. Each F_l will have $(k+2)$ components which will be asymptotic to the $k+2$ lines obtained in the linear class of boundary conditions. This presumably should enable us to associate them with the M label of the boundary states. Thus, the boundary states correspond to *real algebraic curves* in the ϕ-plane whose image in the W-plane are *straight lines* parallel to the $\mathrm{Re}\,W$ axis. In the degenerate case, it is easy to see that all c_L are coincident.

5.2 The general case

We will now consider the general case of a LG model with n chiral superfields and arbitrary superpotential. We will impose n independent conditions

$$F_a(\phi, \overline{\phi}) = 0\,, \tag{5.5}$$

where F_a are real functions. We will use the indices $i, j, \ldots$ to denote the superfields and the indices $a, b, c, \ldots$ to indicate the boundary conditions. Let Σ denote the sub-manifold in $\mathbb{C}^n$ (with complex coordinates ϕ_i and $\overline{\phi}$) obtained by imposing these conditions. We will in addition require that the functions be compatible:

$$\left\{F_a\left(\phi, \overline{\phi}\right), F_b\left(\phi, \overline{\phi}\right)\right\}_{PB} = 0\,. \tag{5.6}$$

We will assume that for all point on Σ, the normals $\vec{n}_a \equiv (\partial_i F_a, \overline{\partial}_i F_a)$ span the normal bundle $\mathcal{N}\Sigma$. The vanishing of the Poisson bracket can be rewritten as

$$\vec{n}_a \cdot \vec{t}_b = 0\,, \tag{5.7}$$

[6]This condition is motivated by the simple observation that an open-string in its ground state can be considered to be a worldsheet soliton (i.e. a kink interpolating between two vacua). Given that the boundary conditions preserve half of the worldsheet supersymmetry, the lowest energy configuration of an open-string (subject to such boundary conditions) is given by a worldsheet BPS condition [23]. For A-type boundary conditions, these are real algebraic curves in the ϕ-plane passing through the vacua. See [24] for a useful discussion.

where $\vec{t}_b \equiv (\partial_i F_a, -\overline{\partial}_i F_a)$ are tangent vectors to the curve $F_b = 0$. It follows that they span the tangent bundle $T\Sigma$. Thus, Σ is a *lagrangian submanifold* of $\mathbb{C}^n$ by construction [11]. The induced metric (first fundamental form) on Σ is given by

$$h_{ab} = \vec{t}_a \cdot \vec{t}_b = \vec{n}_a \cdot \vec{n}_b \,. \tag{5.8}$$

Let h^{ab} denote the inverse of the metric.

The first supersymmetric variation of the boundary conditions leads to

$$\frac{\partial F_a}{\partial \phi_i}\ \psi_{+i} + \eta\ \frac{\partial F_a}{\partial \overline{\phi}_i}\ \overline{\psi}_{-i} = 0\,, \tag{5.9}$$

where the complex conjugate conditions are implicitly assumed. Defining

$$\chi_{\pm a} \equiv \frac{\partial F_a}{\partial \phi_i}\ \psi_{\pm i}\,,$$

the above condition takes the simple form

$$\chi_{+a} + \eta \overline{\chi}_{-a} = 0\,. \tag{5.10}$$

Further supersymmetric variation of the above condition gives rise to the following terms after imposing $\epsilon_+ = \eta\overline{\epsilon}_-$

$$\begin{aligned} &\frac{\partial^2 F_a}{\partial\phi_i\partial\phi_j}\left[-\sqrt{2}\epsilon_-\psi_{+j}+\sqrt{2}\eta\overline{\epsilon}_-\psi_{-j}\right]\psi_{+i} + \frac{\partial^2 F_a}{\partial\phi_i\partial\overline{\phi}_j}\left[\sqrt{2}\overline{\epsilon}_-\overline{\psi}_{+j} + \sqrt{2}\eta\epsilon_-\overline{\psi}_{-j}\right]\psi_{+i}+\\ &+\frac{\partial F_a}{\partial\phi_i}\left[i\sqrt{2}(\partial_0+\partial_1)\phi_i\overline{\epsilon}_-+\sqrt{2}\eta\overline{\epsilon}_-\frac{\partial\overline{W}}{\partial\overline{\phi}_i}\right]+\eta\frac{\partial^2 F_a}{\partial\overline{\phi}_i\partial\overline{\phi}_j}\left[\sqrt{2}\overline{\epsilon}_-\overline{\psi}_{+j}+\sqrt{2}\eta\epsilon_-\overline{\psi}_{-j}\right]\overline{\psi}_{-i}+\\ &+\eta\frac{\partial^2 F_a}{\partial\overline{\phi}_i\partial\phi_j}\left[-\sqrt{2}\epsilon_-\psi_{+j}+\sqrt{2}\eta\overline{\epsilon}_-\psi_{-j}\right]\overline{\psi}_{-i}+\eta\frac{\partial F_a}{\partial\overline{\phi}_i}\left[i\sqrt{2}(\partial_0-\partial_1)\overline{\phi}_i\eta\overline{\epsilon}_-+\sqrt{2}\overline{\epsilon}_-\frac{\partial W}{\partial\phi_i}\right]=0. \end{aligned} \tag{5.11}$$

In the above, the terms involving $\partial_0\phi$ can be seen to vanish using $\partial_0 F = 0$. The remaining terms can be rearranged in an elegant fashion using the extrinsic curvature tensor K_{abc} as defined in the appendix. After eliminating χ_{+a} and $\overline{\chi}_{+a}$, respectively in terms of $\overline{\chi}_{-a}$ and χ_{-a}, we obtain

$$\begin{aligned} &\epsilon_- K_{abc}\ \chi^b_-\ \chi^c_-\ + \overline{\epsilon}_-\left(+i\left(\frac{\partial F_a}{\partial\phi_i}\partial_1\phi_i - \frac{\partial F_a}{\partial\overline{\phi}_i}\partial_1\overline{\phi}_i\right) + K_{abc}\ \chi^b_-\ \overline{\chi}^c_-\right) +\\ &\qquad + \ \eta\overline{\epsilon}_-\left(\frac{\partial F_a}{\partial\phi_i}\frac{\partial\overline{W}}{\partial\overline{\phi}_i} + \frac{\partial F_a}{\partial\overline{\phi}_i}\frac{\partial W}{\partial\phi_i}\right) = 0\,, \end{aligned} \tag{5.12}$$

where $\chi^a_- = h^{ab}\chi_{-b}$. In the above terms, it can be seen that the term multiplying ϵ_- cancels since a symmetric object multiplies an antisymmetric object. The terms multiplying $\overline{\epsilon}_-$ lead to two new conditions rather than a single one. There are two ways to understand this: first, the terms involving the superpotential are multiplied with an η and if we insist on a single condition, the bosonic boundary condition ends up depending on the spin structure. In addition, the vanishing of the boundary terms in the ordinary variation also requires two conditions. The two conditions are

$$\begin{aligned}\left(\left[\frac{\partial F_a}{\partial \phi_i}\partial_1\phi_i - \frac{\partial F_a}{\partial \overline{\phi}_i}\partial_1\overline{\phi}_i\right] - iK_{abc}\,\chi^b_-\ \overline{\chi}^c_-\right) &= 0\,,\\ \left\{F_a(\phi,\overline{\phi}), W(\Phi) - \overline{W}(\overline{\phi})\right\}_{PB} &= 0\,.\end{aligned} \tag{5.13}$$

We note that the demonstration of the cancellation of the boundary terms of the ordinary and supersymmetric variation of the action is tedious but straightforward. The full set of boundary conditions obtained by the the requirement of unbroken $N=2$ supersymmetry of the A-type is equivalent to requiring that the submanifold Σ be *lagrangian*. For the case without a superpotential, this corresponds to the microscopic(worldsheet) realisation of situations considered by Harvey and Lawson [11]. The *new feature* that we obtain is that in the presence of a superpotential, there is an additional condition that the real conditions F_a have a vanishing Poisson bracket with $(W-\overline{W})$. This suggests that one must *necessarily* choose one of the conditions to be $F=(W-\overline{W})-ic$ where c is a real constant. This can be seen as a consequence of the fact that in a phase space of real dimension $2n$, there can only n independent commuting constants of motion.

5.3 Spacetime supersymmetry and the special lagrangian condition

The special lagrangian condition[7] which is necessary for spacetime supersymmetric D-brane configuration appears in our microscopic description as follows. (We will first discuss the case when there is no superpotential. This is the case where the spacetime is $\mathbb{C}^n$). Let Υ and $\overline{\Upsilon}$, respectively be the holomorphic $(n,0)$ form and anti-holomorphic $(0,n)$ form on $\mathbb{C}^n$. In the microscopic description, we can choose

$$\begin{aligned}\Upsilon &\equiv \psi_{-1}\psi_{-2}\cdots\psi_{-n}\,,\\ \overline{\Upsilon} &\equiv \overline{\psi}_{+1}\overline{\psi}_{+2}\cdots\overline{\psi}_{+n}\,.\end{aligned} \tag{5.14}$$

This choice is dictated by the fact that under the A-twist, ψ_{-i} become $(1,0)$ forms and $\overline{\psi}_{+i}$ become $(0,1)$ forms on $\mathbb{C}^n$. One can also see (by bosonising the fermions, for example) that Υ generates spectral flow in the left-moving $N=2$ under which the NS and R sectors get mapped to each other. $\overline{\Upsilon}$ has a similar action in the right moving sector except that it is a spectral flow of opposite sign. The requirements for having boundary conditions on the worldsheet which preserve spacetime supersymmetry are:

(i) The boundary conditions must preserve a global $N=2$ worldsheet supersymmetry.

[7] This was derived for the first time using spacetime supersymmetry by ref. [25]. Of the subsequent literature on this approach, the one closest in spirit to our microscopic viewpoint is ref. [26].

(ii) The boundary conditions must preserve a linear combination of the two spectral flow generators [1].

In our earlier considerations, we have ensured that the first part has been satisfied. The second condition can be stated as follows:

$$\Upsilon = \eta^n \exp(i\alpha)\overline{\Upsilon}\,, \tag{5.15}$$

for some constant α. Under the general A-type boundary conditions discussed above, one can see that (using eq. (5.9))

$$\Delta\ \Upsilon \equiv \Delta\ \psi_{-1}\psi_{-2}\cdots\psi_{-n} = (-)^n\overline{\Delta}\ \overline{\psi}_{+1}\overline{\psi}_{+2}\cdots\overline{\psi}_{+n} = (-)^n\overline{\Delta}\ \overline{\Upsilon}\,, \tag{5.16}$$

where $\Delta \equiv \text{Det}\ \frac{\partial F_a}{\partial \phi_i}$. The special lagrangian condition can now be restated as

$$\Delta = (-)^n \exp(i\alpha)\ \overline{\Delta}\,. \tag{5.17}$$

The $(-)^n$ can always be absorbed into the phase α and thus is not crucial. Hence the general boundary conditions which preserve spacetime supersymmetry have to satisfy eq. (5.17).

So far we have discussed special lagrangian condition for the case when the superpotential was zero, i.e. the target space was $\mathbb{C}^n$. In the presence of a superpotential, we have seen that it is necessary to choose one of the conditions, say

$$F_1 = W(\phi) - \overline{W}(\overline{\phi})\,.$$

Further, let us assume that the superpotential is homogeneous and that ϕ_i are homogeneous coordinates on $\mathbb{P}^{n-1}$.[8] In order that the boundary conditions carry over to $\mathbb{P}^{n-1}$, we will require that the F_a be homogeneous under real scalings: $\phi_i \to \lambda\phi_i$, for real $\lambda \neq 0$. Clearly, this is satisfied by $F_1 = W(\phi) - \overline{W}(\overline{\phi})$.

Suppose, we have chosen F_a which satisfy the conditions mentioned in the previous paragraph and that the special lagrangian condition in $\mathbb{C}^n$ given in eq. (5.17) is also satisfied. We will now show that this implies that one obtains a special lagrangian submanifold Σ of the Calabi-Yau manifold described by the equation $W = 0$ in $\mathbb{P}^{n-1}$. The global holomorphic $(n-2,0)$ form on the Calabi-Yau manifold is given by [27, 28]

$$\Omega = \int_\gamma \frac{\epsilon^{i_1\cdots i_n}\phi_{i_1} d\phi_{i_2}\cdots d\phi_{i_n}}{W(\phi)}\,, \tag{5.18}$$

where γ is a curve in $\mathbb{P}^{n-1}$ enclosing $W = 0$. On Σ, the F_a satisfy

$$dF_a = \frac{\partial F_a}{\partial \phi_i}\ d\phi_i + \frac{\partial F_a}{\partial \overline{\phi}_i} d\overline{\phi}_i = 0\,. \tag{5.19}$$

[8]The generalisation to the case of hypersurfaces in weighted projective spaces is obvious. We shall restrict to projective spaces for simplicity.

Further, the homogeneity condition on the boundary conditions F_a can be written as

$$\phi_i \frac{\partial F_a}{\partial \phi_i} + \overline{\phi}_i \frac{\partial F_a}{\partial \overline{\phi}_i} = d_a F_a \,, \tag{5.20}$$

where d_a is the degree of F_a. Using the above two relations and the fact that we choose $W(\phi) = \overline{W}(\overline{\phi})$ as one of our boundary conditions, one can see that on Σ

$$\epsilon^{i_1 \cdots i_n} \phi_{i_1} d\phi_{i_2} \cdots d\phi_{i_n} \big|_\Sigma = \frac{\overline{\Delta}}{\Delta} \epsilon^{j_1 \cdots j_n} \overline{\phi}_{j_1} d\overline{\phi}_{j_2} \cdots d\overline{\phi}_{j_n} \Big|_\Sigma \,. \tag{5.21}$$

This implies that

$$\Omega|_\Sigma = (-)^n \frac{\overline{\Delta}}{\Delta} \, \overline{\Omega}|_\Sigma = \exp(-i\alpha) \, \overline{\Omega}|_\Sigma \,, \tag{5.22}$$

which is the special lagrangian condition on the Calabi-Yau manifold. Note that this has not been derived using any spacetime inputs but rather from the worldsheet analysis of the LG model with boundary.

An important question to consider is whether the homogeneity condition, eq. (5.20), which certainly appears natural, is too restrictive. One possibility is to allow for the condition

$$\left(\phi_i \frac{\partial F_a}{\partial \phi_i} + \exp(i\theta) \overline{\phi}_i \frac{\partial F_a}{\partial \overline{\phi}_i} \right) \Bigg|_\Sigma = 0 \bmod W \,, \tag{5.23}$$

where θ is a constant. The mod$W$ degree of freedom reflects the fact that the integral in eq. (5.18) has support only at the zeros of W. We will use this weaker condition shortly in an example. With this weaker condition, one obtains

$$\Omega|_\Sigma = \exp(-i\alpha + i\theta) \overline{\Omega}|_\Sigma \,. \tag{5.24}$$

5.4 Examples

We first illustrate the case without a superpotential by using the classic example of Harvey and Lawson [11]. The construction provides special lagrangian submanifolds with topology $\mathbb{R}^+ \times \mathbb{T}^{n-1}$ on $\mathbb{C}^n$. We will also show that this leads naturally to a T^{n-1} lagrangian fibration of $\mathbb{P}^{n-1}$.

The conditions of Harvey and Lawson can be implemented as boundary conditions in our worldsheet theory:

$$\begin{aligned} F_1 &= \begin{cases} \mathrm{Re}(\phi_1 \cdots \phi_n) - c_1 & \text{for even } n \\ \mathrm{Im}(\phi_1 \cdots \phi_n) - c_1 & \text{for odd } n \,, \end{cases} \\ F_a &= |\phi_1|^2 - |\phi_a|^2 - c_a \quad \text{for } 2 \le a \le n \,, \end{aligned} \tag{5.25}$$

where c_a are some real constants. Linear combinations of the c_a correspond to the radii of the circles of $\mathbb{T}^{n-1}$. When $c_1 = 0$, the $\mathbb{R}^+$ corresponds to the value of $|\phi_1|^2$ and the $\mathbb{T}^{n-1}$ corresponds to the $(n-1)$ phases left unfixed by the condition $F_1 = 0$.

In order to extend these boundary conditions to $\mathbb{P}^{n-1}$, the condition of homogeneity on the F_a implies that all the constants c_a must be necessarily set to zero. In this limit, the lagrangian submanifold appears to be singular at the origin. This however is not a point in $\mathbb{P}^{n-1}$. Thus, the submanifold is non singular. Further, the (real) scaling degree of freedom in the homogeneous coordinates ϕ_i of $\mathbb{P}^{n-1}$ eats up the $\mathbb{R}^+$ degree of freedom leaving us with a $\mathbb{T}^{n-1}$. In the inhomogeneous coordinates of $\mathbb{P}^{n-1}$, where we set $\phi_1 = 1$, the radii of all circles is set to unity. Thus the lagrangian submanifold, $\mathbb{T}^{n-1}$ in $\mathbb{P}^{n-1}$ is obtained at a specific point in its moduli space. We will momentarily see how to generalise this.

As pointed out by Strominger, Yau and Zaslow [29], the existence of a mirror partner for a Calabi-Yau threefold implies that the Calabi-Yau manifold admits a $\mathbb{T}^3$ fibration. Consider the mirror quintic given by the equation in $\mathbb{P}^4$:

$$W = \phi_1^5 + \cdots + \phi_5^5 - 5\psi\phi_1\phi_2\phi_3\phi_4\phi_5 = 0 \,. \tag{5.26}$$

The large complex structure limit corresponds to $|\psi| \to \infty$. In the infinite limit, the quintic breaks up into five $\mathbb{P}^3$ given by setting $\phi_i = 0$ for $i = 1, \ldots, 5$. While this is a degenerate limit, the $\mathbb{T}^3$ special lagrangian fibre is seen easily by using the earlier construction for $\mathbb{P}^3$. It has been argued by Strominger et al., that this $\mathbb{T}^3$ will be special lagrangian in the neighbourhood of the infinite $|\psi|$ limit(see ref. [30] for a discussion).

Since it in general rather hard to construct special lagrangian submanifolds, it is of interest to see how the above example fits into our construction. In the infinite complex structure limit, it is interesting to note that F^1 chosen by Harvey and Lawson is indeed equal to $(W - \overline{W})$! It immediately follows therefore that if we also choose the conditions

$$F_a = |\phi_1|^2 - |\phi_a|^2 - c_a \quad \text{for } 2 \leq a \leq 5 \tag{5.27}$$

then we obtain a supersymmetric cycle. We have introduced four constants c_a which break the homogeneity condition. However, one can check that the weaker condition mentioned earlier holds:

$$\left.\left(\phi_i \frac{\partial F_a}{\partial \phi_i} - \overline{\phi}_i \frac{\partial F_a}{\partial \overline{\phi}_i}\right)\right|_\Sigma = 0 \bmod W \,. \tag{5.28}$$

A calculation shows that except for $F_1 = (W - \overline{W})$, the above condition holds identically (without the mod$W$ condition). It is of interest to count the number of independent parameters. First, it may seem that we have four angles and thus a $\mathbb{T}^4$. However, since $W = 0$ necessarily requires one of the ϕ_i to identically vanish, the angle associated with the vanishing ϕ_i does not exist. Further, projectivisation leaves only three independent real variables coming from the four c_a. These presumably correspond to the moduli associated with the $\mathbb{T}^3$. The example discussed above is not something specific to the quintic but can be extended to a larger class of CY threefolds. See for instance, ref. [31].

6. Conclusions and outlook

It is clear that the methods that we have outlined in this paper have obvious generalizations. First, the methods can be extended to LG models that are associated to CY hypersurfaces in weighted projective spaces. Second, since we are working in the LG model we can extend our techniques to hypersurfaces that are described by more general potentials than those of the Fermat type. Thirdly, the techniques could equally well be applied in the case of non-quasi-homogeneous potentials relevant to massive $N = 2$ supersymmetric theories. There would however be some difference in the geometric interpretation of the boundary conditions in these cases.

The special lagrangian submanifolds considered here, described by a set of real equations $F_a = 0$ in some ambient $\mathbb{C}^n$ can be thought of as a real algebraic variety. This is in line with the theory in the bulk corresponding to strings propagating on a complex algebraic variety. It would be interesting to see how other structures that were seen in the bulk, like the operator ring for instance, carry over to the boundary theory. It is not clear however what the potential role, if any at all, of anholonomic boundary conditions, which of course are allowed in general. While inequalities together with conditions of the form $F_a = 0$ are in general relevant to what are known to mathematicians as semi-algebraic real sets. It would be interesting to see if they have a role in the context of special lagrangian sub-manifolds in CY.

One of the themes of this paper was to obtain boundary conditions corresponding to all the boundary states in a single minimal model. While the non-linear boundary conditions we have constructed have suggested a possible way out, the story is far from complete. It is of interest to be able to compute the Witten index for the non-linear case in order to be sure of the identification proposed in this paper. Solving this problem is of interest in making a clear geometric identification of the Recknagel-Schomerus class of boundary states. In particular, we do not yet have a clear geometric identification of all the $L \neq \lfloor k/2 \rfloor$ states in the boundary CFT. We may add that even in the linear class of boundary conditions we have not yet explored the cases where the matrix A is non diagonal. This should help us in examining boundary states in the Gepner construction that do not belong to the Recknagel-Schomerus class of states. Such states naturally arise from the possible fixed-point resolutions of the modular transformation matrix of the full Gepner CFT [5].

It is of interest to extend our analysis to the case of B-type boundary conditions. However, the LG description of B-branes will be rather different from the large volume CY description since the geometry and charge of the B-branes are not expected to remain invariant. Nevertheless, it may be possible to track some B-branes from the large volume CY limit to the LG phase without encountering lines of marginal stability. Assuming that this is possible, then one might be able to calculate the worldvolume superpotential directly in the LG model.

As we discussed in section 2, it is not possible to impose Neumann boundary conditions on all fields in the LG model. It is also not possible to impose Neumann boundary conditions on the LG field of a single minimal model. This strongly suggests that all the states of the Recknagel-Schomerus class must arise from Dirichlet-type boundary conditions in the LG and suitable modifications thereof. This shows that for example, a D6-brane wrapping a CY will look rather different in the LG limit. From the work of Brunner et al. we know that the corresponding state exists at the Gepner point in the moduli space. It would be of interest to describe this state in the LG formalism. Given the identification of the linear LG boundary conditions of A-type with the $L = 1$ boundary states of the A-type boundary CFT, the case of B-type Dirichlet boundary conditions on all LG fields described in section 2, most likely are the $\{L_i = 1,\ \text{for all}\ i\}$ B-type states.

The B-type states may also be studied by using mirror symmetry on A-type states in the LG model. In the LG models, the orbifolding technique provides a simple method of constructing the mirror CY and should also provide a corresponding method for the construction of A-type boundary states in the mirror.

Given the close interplay of both complex and Kähler moduli in the description of D-branes on CY threefolds (a natural consequence of spacetime $N = 2$ supersymmetry being broken to $N = 1$ by the D-branes), the linear sigma model(LSM) description is better suited in some ways for a microscopic description on CY threefolds. For example, one can show that the D6-brane (all Neumann boundary conditions) in the CY limit starts looking like an all Dirichlet boundary condition as one goes to "small volumes" and to the LG phase. In the neighbourhood of vanishing Kähler parameter (for the quintic), the CY as seen by the D-brane appears to be in a non-commutative phase. These issues will be discussed in a forthcoming paper. Related remarks appear in the work of Hori and Vafa [14]. The transitions discussed by Joyce [32] also seem well suited for an LSM description.

Acknowledgments

We would like to thank the following for useful discussions: R. Balakrishnan, V. Balakrishnan, G. Date, S. Lakshmi Bala, D.S. Nagaraj, K. Paranjape and especially T. Sarkar. S.G. is supported in part by the Department of Science and Technology, India under the grant SP/S2/E-03/96.

A. Extrinsic curvature of lagrangian submanifolds

We will be considering a *lagrangian submanifold* Σ of $\mathbb{C}^n$ which is implicitly specified by n independent *real* functions

$$F_a(\phi, \overline{\phi}) = 0\,,$$

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where ϕ_i are complex coordinates on $\mathbb{C}^n$. The lagrangian condition implies that the Poisson bracket of the n functions vanish [11]. Further, the normals $n^i_a = (\partial_i F_a, \overline{\partial}_i F_a)$ span the normal bundle $\mathcal{N}\Sigma$ and the tangents $t^i_a = (\partial_i F_a, -\overline{\partial}_i F_a)$ span the tangent bundle $T\Sigma$ and $T\mathbb{C}^n = \mathcal{N}\Sigma \oplus T\Sigma$. The vanishing Poisson bracket ensures that $\vec{n}_a \cdot \vec{t}_b = 0$.

The tangential derivatives $D_a \equiv t^i_a \partial_i$ satisfy $[D_a, D_b] = 0$ by virtue of the vanishing of the Poisson bracket $\{F_a, F_b\}_{PB} = 0$. Thus, locally on Σ, there exists a coordinate system σ_a such that $\partial/\partial\sigma_a = D_a$. The induced metric(*first fundamental form*) in this coordinate system is given by

$$h_{ab} = \vec{t}_a \cdot \vec{t}_b = \vec{n}_a \cdot \vec{n}_b \,. \tag{A.1}$$

The extrinsic curvature tensor (*second fundamental form*) $\vec{K}_{ab}$ is defined as follows[9]

$$\left(t^i_a \partial_i\right)\ t^j_b = \left(t^i_b \partial_i\right)\ t^j_a = K^j_{ab} + \Gamma^c_{ab}\ t^j_c \,, \tag{A.2}$$

where Γ^c_{ab} is the Christoffel connection with respect to the induced metric on Σ. Thus, $\vec{K}_{ab}$ is normal to the surface Σ (since the second term projects out the tangential component of $(t^i_a\partial_i)t^j_b$). Since, $\vec{n}_a$ span the $\mathcal{N}\Sigma$, we can decompose $\vec{K}_{ab}$ into

$$K_{abc} \equiv \vec{K}_{ab} \cdot \vec{n}_c \,. \tag{A.3}$$

One can explicitly calculate K_{abc} as defined above and we obtain

$$\begin{aligned} K_{abc} = - &\left[\frac{\partial F_c}{\partial\phi_i}\frac{\partial F_b}{\partial\phi_j}\frac{\partial^2 F_a}{\partial\phi_i\partial\phi_j} - \frac{\partial F_c}{\partial\overline{\phi}_i}\frac{\partial F_b}{\partial\phi_j}\frac{\partial^2 F_a}{\partial\phi_i\partial\overline{\phi}_j} - \frac{\partial F_c}{\partial\phi_j}\frac{\partial F_b}{\partial\overline{\phi}_i}\frac{\partial^2 F_a}{\partial\phi_i\partial\overline{\phi}_j} + \right. \\ &\left. + \frac{\partial F_c}{\partial\overline{\phi}_i}\frac{\partial F_b}{\partial\overline{\phi}_j}\frac{\partial^2 F_a}{\partial\overline{\phi}_i\partial\overline{\phi}_j} \right] . \end{aligned} \tag{A.4}$$

One can verify, that K_{abc} is a completely symmetric tensor. The symmetry under the exchange $a \leftrightarrow b$ is the usual symmetry property of the extrinsic curvature tensor. However, for *lagrangian submanifolds*, one has the isomorphism between the normal bundle and the tangent bundle which enables one to make it fully symmetric [34]. Further, if Σ is special lagrangian, then the trace of the extrinsic curvature tensor with respect to the induced metric vanishes.

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